Cooperative Games

Lecture 7: The Shapley Value

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The Shapley value

Lloyd S. Shapley. **A Value for n-person Games.** In *Contributions to the Theory of Games, volume II (Annals of Mathematical Studies)*, 1953.

Definition (marginal contribution)

The **marginal contribution** of agent i for a coalition $\mathcal{C} \subseteq N \setminus \{i\}$ is $mc_i(\mathcal{C}) = v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})$.

$\langle mc_1(\emptyset), mc_2(\{1\}), mc_3(\{1,2\})\rangle$ is an efficient payoff distribution for any game $(\{1,2,3\}, v)$. This payoff distribution may model a dynamic process in which 1 starts a coalition, is joined by 2, and finally 3 joins the coalition $\{1,2\}$, and where the incoming agent gets its marginal contribution.

An agent's payoff depends on which agents are already in the coalition. This payoff may not be **fair**. To increase fairness,one could take the average marginal contribution over all possible joining orders.

Let σ represent a joining order of the grand coalition N, i.e., σ is a permutation of $\langle 1, \dots, n\rangle$.
We write $mc(\sigma) \in \mathbb{R}^n$ the payoff vector where agent i obtains $mc_i(\{\sigma(j) \mid j < i\})$. The vector mc is called a **marginal vector**.

Shapley value: version based on marginal contributions

Let (N,v) be a TU game. Let $\Pi(N)$ denote the set of all permutations of the sequence $\langle 1, \dots, n\rangle$.

$$Sh(N,v) = \frac{\sum_{\sigma \in \Pi(N)} mc(\sigma)}{n!}$$

the Shapley value is a **fair** payoff distribution based on marginal contributions of agents averaged over joining orders of the coalition.

An example

$N = \{1,2,3\}$, $v(\{1\}) = 0$, $v(\{2\}) = 0$, $v(\{3\}) = 0$,
$v(\{1,2\}) = 90$, $v(\{1,3\}) = 80$, $v(\{2,3\}) = 70$,
$v(\{1,2,3\}) = 120$.

	1	2	3
$1 \leftarrow 2 \leftarrow 3$	0	90	30
$1 \leftarrow 3 \leftarrow 2$	0	40	80
$2 \leftarrow 1 \leftarrow 3$	90	0	30
$2 \leftarrow 3 \leftarrow 1$	50	0	70
$3 \leftarrow 1 \leftarrow 2$	80	40	0
$3 \leftarrow 2 \leftarrow 1$	50	70	0
total	270	240	210
Shapley value	45	40	35

Let $y = \langle 50, 40, 30\rangle$

$\mathcal{C}$	$e(\mathcal{C},x)$	$e(\mathcal{C},y)$
{1}	-45	0
{2}	-40	0
{3}	-35	0
{1,2}	5	0
{1,3}	0	0
{2,3}	-5	0
{1,2,3}	120	0

This example shows that the Shapley value may not be in the core, and may not be the nucleolus.

- There are $|\mathcal{C}|!$ permutations in which all members of $\mathcal{C}$ precede i.
- There are $|N \setminus (\mathcal{C} \cup \{i\})|!$ permutations in which the remaining members succede i, i.e. $(|N| - |\mathcal{C}| - 1)!$.

The Shapley value $Sh_i(N,v)$ of the TU game (N,v) for player i can also be written

$$Sh_i(N,v) = \sum_{\mathcal{C} \subseteq N \setminus \{i\}} \frac{|\mathcal{C}|!(|N| - |\mathcal{C}| - 1)!}{|N|!} \left(v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})\right).$$

Using definition, the sum is over $2^{|N|-1}$ instead of $|N|!$.

Notion of value

Definition (value function)

Let $\mathcal{G}_N$ the set of all TU games (N,v). A **value function ϕ** is a function that assigns to each TU game (N,v) an efficient allocation, i.e. $\phi : \mathcal{G}_N \to \mathbb{R}^{|N|}$ such that $\phi(N,v)(N) = v(N)$.

- The Shapley value is a value function.
- None of the concepts presented thus far were a value function (the nucleolus is guaranteed to be non-empty only for games with a non-empty set of imputations).

Some interesting properties

Let (N,v) and (N,u) be TU games and ϕ be a value function.

- **Symmetry or substitution (SYM):** If $\forall (i,j) \in N$, $\forall \mathcal{C} \subseteq N \setminus \{i,j\}$, $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$ then $\phi_i(N,v) = \phi_j(N,v)$
- **Dummy (DUM):** If $\forall \mathcal{C} \subseteq N \setminus \{i\}$ $v(\mathcal{C}) + v(\{i\}) = v(\mathcal{C} \cup \{i\})$, then $\phi_i(N,v) = v(\{i\})$.
- **Additivity (ADD):** Let $(N, u+v)$ be a TU game defined by $\forall \mathcal{C} \subseteq N$, $(u+v)(N) = u(N) + v(N)$. $\phi(u+v) = \phi(u) + \phi(v)$.

Theorem

The Shapley value is the unique value function ϕ that satisfies (SYM), (DUM) and (ADD).

Unanimity game

Let N be a set of agents and $T \subseteq N \setminus \emptyset$.
The **unanimity game** (N, v_T) is defined as follows:
$\forall \mathcal{C} \subseteq N,\ v_T(\mathcal{C}) = \begin{cases} 1, \text{ if } T \subseteq \mathcal{C}, \\ 0 \text{ otherwise.} \end{cases}$

We note that

- if $i \in N \setminus T$, i is a null player.
- if $(i,j) \in T^2$, i and j are substitutes.

Lemma

The set $\{v_T \mid T \subseteq N \setminus \emptyset\}$ is a linear basis of $\mathcal{G}_N$.

This means that a TU game (N,v) can be represented by a unique set of values $(\alpha_T)_{T \subseteq N \setminus \emptyset}$ such that
$\forall \mathcal{C} \subseteq N,\ v(\mathcal{C}) = \left(\sum_{T \subseteq N \setminus \emptyset} \alpha_T v_T\right)(\mathcal{C})$.

Proof of the lemma

There are $2^n - 1$ unanimity games and the dimension of $\mathcal{G}_N$ is also $2^n - 1$.
We only need to prove that the unanimity games are linearly independent.
Towards a contradiction, let us assume that $\sum_{T \subseteq N \setminus \emptyset} \alpha_T v_T = 0$ where $(\alpha_T)_{T \subseteq N \setminus \emptyset} \neq 0_{\mathbb{R}^{2^n - 1}}$.
Let T_0 be a minimal set in $\{T \subseteq N \mid \alpha_T \neq 0\}$.
Then, $\left(\sum_{T \subseteq N \setminus \emptyset} \alpha_T v_T\right)(T_0) = \alpha_{T_0} \neq 0$, which is a contradiction.

Proof of the theorem: Uniqueness (1/2)

Let ϕ a feasible solution on $\mathcal{G}_N$ that is non-empty and satisfies the axioms SYM, DUM and ADD. Let us prove that ϕ is a value function.Let $(N,v) \in \mathcal{G}_N$.

- if $v = 0_{\mathcal{G}_N}$, all players are dummy. Since the solution is non-empty, $0^{\mathbb{R}^{|N|}}$ is the unique member of $\phi(N,v)$.
- otherwise, $(N,-v) \in \mathcal{G}_N$.
  Let $x \in \phi(N,v)$ and $y \in \phi(N,-v)$.
  By ADD, $x + y \in \phi(v - v)$, and then, $x = -y$ is unique.
  Moreover, $x(N) \leqslant v(N)$ as ϕ is a feasible solution.
  Also $y(N) \leqslant -v(N)$.
  Since $x = -y$, we have $v(N) \leqslant x(N) \leqslant v(N)$,
  i.e. x is efficient.

Hence, ϕ is a value function.

Proof of the theorem: Uniqueness (2/2)

Let $T \subseteq N \setminus \emptyset$ and $\alpha \in \mathbb{R}$. Let us prove that $\phi(N, \alpha \cdot v_T)$ is uniquely defined.

- Let $i \notin T$. We have trivially $T \subseteq \mathcal{C}$ iff $T \subseteq \mathcal{C} \cup \{i\}$. Then $\forall \mathcal{C} \subseteq N \setminus \{i\}$, $\alpha v_T(\mathcal{C}) = \alpha v_T(\mathcal{C} \cup \{i\})$.Hence, all agent $i \notin T$ are dummies. By DUM, $\forall i \notin T$, $\phi_i(N, \alpha \cdot v_T) = 0$.
- Let $(i,j) \in T^2$.Then for all $\mathcal{C} \subseteq N \setminus \{i,j\}$, $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$.By SYM, $\phi_i(N, \alpha \cdot v_T) = \phi_j(N, \alpha \cdot v_T)$.
- Since ϕ is a value function, it is efficient. Then, $\sum_{i \in N} \phi_i(N, \alpha \cdot v_T) = \alpha v_T(N) = \alpha$.
  Hence, $\forall i \in T$, $\phi_i(N, \alpha \cdot v_T) = \frac{\alpha}{|T|}$.

This proves that $\phi(N, \alpha \cdot v_T)$ is uniquely defined. Since any TU game (N,v) can be written as $\sum_{T \subseteq N \setminus \emptyset} \alpha_T v_T$ and because of ADD, there is a unique value function that satisfies the three axioms.

Proof of the theorem: Existence

We need to show that the Shapley value satisfies the three axioms. Let (N,v) a TU game.

$$Sh(N,v) = \frac{\sum_{\sigma \in \Pi(N)} mc(\sigma)}{n!}$$

- Let us assume that $\forall \mathcal{C} \subseteq N \setminus \{i,j\}$, we have $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$. Then $\forall \mathcal{C} \subseteq N \setminus \{i,j\}$, we have
  - $mc_i(\mathcal{C}) = mc_j(\mathcal{C})$
  - $v(\mathcal{C} \cup \{i,j\}) - v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{i,j\}) - v(\mathcal{C} \cup \{j\})$, hence, we have $mc_j(\mathcal{C} \cup \{j\}) = mc_i(\mathcal{C} \cup \{i\})$.
  - $\leadsto$ $Sh_i(N,v) = Sh_j(N,v)$, Sh satisfies SYM.
- Let us assume there is an agent i such that for all $\mathcal{C} \subseteq N \setminus \{i\}$ we have $v(\mathcal{C}) = v(\mathcal{C} \cup \{i\})$.Then, each marginal contribution of player i is zero, and it follows that $Sh_i(N,v) = 0$. Sh satisfies DUM.
- Sh is clearly additive.

✔

Discussion about the axioms

- SYM: it is desirable that two subsitute agents obtain the same value ✔
- DUM: it is desirable that an agent that does not bring anything in the cooperation does not get any value. ✔
- What does the addition of two games mean?
  - \+ if the payoff is interpreted as an expected payoff, ADD is a desirable property.
  - \+ for cost-sharing games, the interpretation is intuitive: the cost for a joint service is the sum of the costs of the separate services.
  - \- there is no interaction between the two games.
  - \- the structure of the game $(N, v + w)$ may induce a behavior that has may be unrelated to the behavior induced by either games (N,v) or (N,w).
- The axioms are independent. If one of the axiom is dropped, it is possible to find a different value function satisfying the remaining two axioms.

Let (N,v) and (N,v) be two TU games.

- **Marginal contribution:** A value function ϕ satisfies marginal contribution axiom iff for all $i \in N$,
  if for all $\mathcal{C} \subseteq N \setminus \{i\}$ $v(\mathcal{C} \cup \{i\}) - v(C) = u(\mathcal{C} \cup \{i\}) - u(C)$,
  then $\phi(u) = \phi(v)$.

The value of a player depends only on its marginal contribution.

Theorem (H.P. Young)

The Shapley value is the unique value function that satisfies symmetry and marginal contribution axioms.

We refer by $v \setminus i$ the TU game $(N \setminus \{i\}, v_{\setminus i})$ where $v_{\setminus i}$ is the restriction of v to $N \setminus \{i\}$.

- **Balanced contribution:** A value function ϕ satisfies balanced contribution iff for all $(i,j) \in N^2$
  $\phi_i(v) - \phi_i(v \setminus j) = \phi_j(v) - \phi_j(v \setminus i)$.

For any two agents, the amount that each agent would win or lose if the other "leaves the game" should be the same.

Theorem (R Myerson)

The Shapley value is the unique value function that satisfies the balanced contribution axiom.

Some properties

Theorem

For superadditive games, the Shapley value is an imputation.

Lemma

For convex game, the Shapley value is in the core.

Proofs

- Let (N,v) be a superadditive TU game.
  By superadditivity, $\forall i \in N$, $\forall \mathcal{C} \subseteq N \setminus \{i\}$
  $v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) > v(\{i\})$. Hence, for each marginal vector, an agent i gets at least $v(\{i\})$. The same is true for the Shapley value as it is the average over all marginal vectors.
- Let (N,v) be a convex game.
  We know that all marginal vectors are in the core (to show that convex games have non-empty core, we used one marginal vector and showed it was in the core).
  The core is a convex set.
  The average of a finite set of points in a convex set is also in the set.
  Finally, the Shapley value is in the core.

Summary

- **pros**
  - The Shapley value is a value function, i.e., it **always exists** and is **unique**.
  - When the valuation function is **superadditive**, the Shapley value is **individually rational**, i.e., it is an imputation.
  - When the valuation function is **convex**, the Shapley value is also group rational, hence, it is in the **core**.
  - The Shapley value is the unique value function satisfying some axioms.
- **cons**
  - The nature of the Shapley value is combinatorial.

Coming next

- Voting games and power indices.